Circle Squaring with Pieces of Small Boundary and Low Borel Complexity

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Joint work with András Máthé and Jonathan A. Noel

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Introducion

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  st $\forall\, i\ \ \exists$ isometry γ_i with $B_i = \gamma_i(A_i)$

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- Marks-Unger'16: simpler proof

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- Open: Analogous results for the Borel σ-algebra $\mathcal{B}$

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  - Laczkovich'90: YES, using translations only

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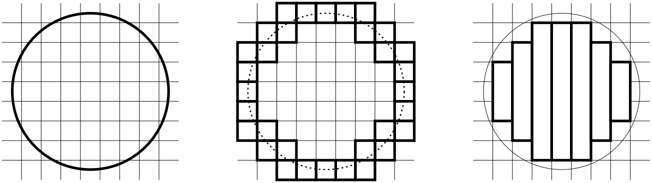

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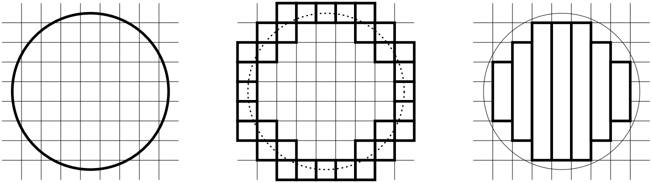

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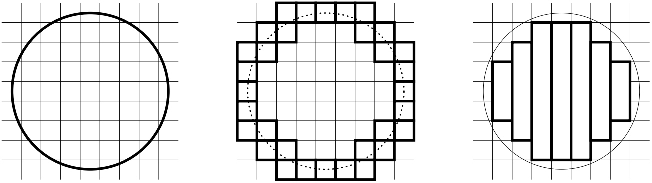

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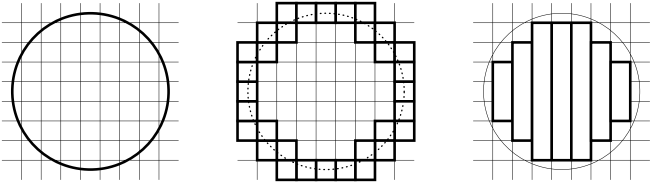

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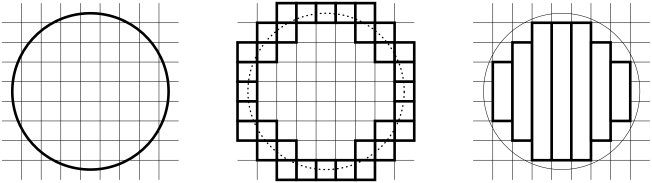

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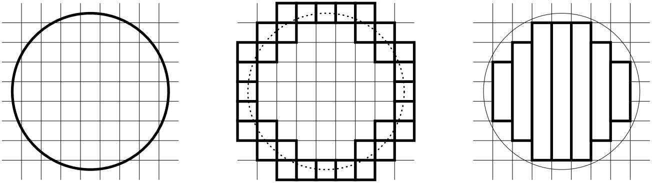

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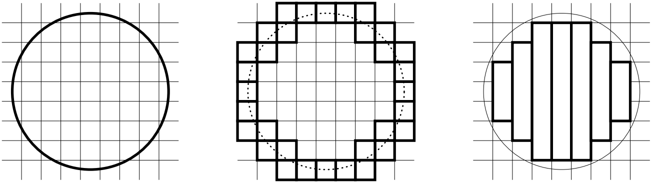

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  - $\exists K \subseteq X \subseteq U$ with $\lambda(U \setminus K) < \varepsilon$, using $O(\varepsilon^{c-k})$ boxes

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- $\Rightarrow$ Circle squaring with $\dim_\square \partial A_i < 1.987$ and $A_i \in \boldsymbol{B}(F_\sigma\text{-sets})$

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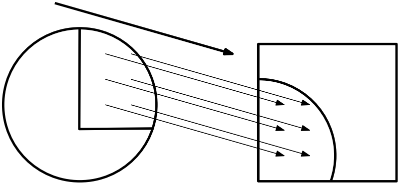

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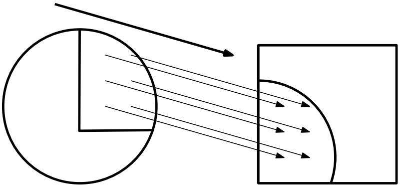

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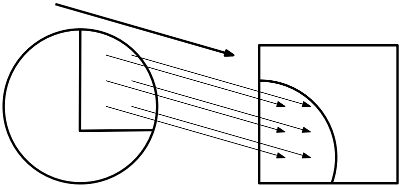

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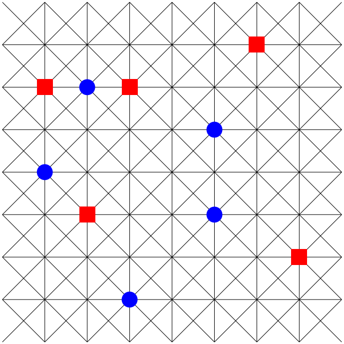
Local picture for $d = 2$ and $M = 2$



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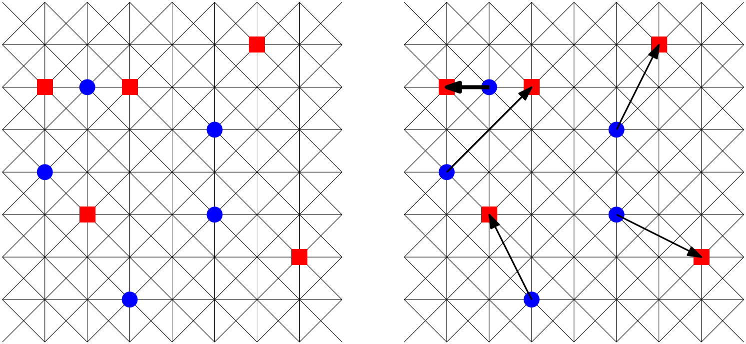

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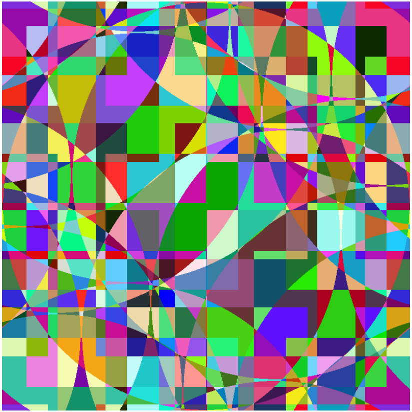

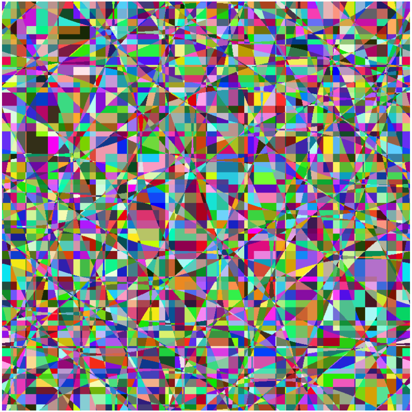

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- A_i, B_i are iM-local functions of A and B

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- Finally: $A_j := \cup_{i=1}^{\infty} A_i^j$
- Lemma: If $A \in \mathcal{J}$, each $A_j^i \in \mathcal{J}$ and $\lambda(A \setminus \cup_{i=1}^{\infty} A_i) = 0$ then $A_1, \dots, A_n \in \mathcal{J}$

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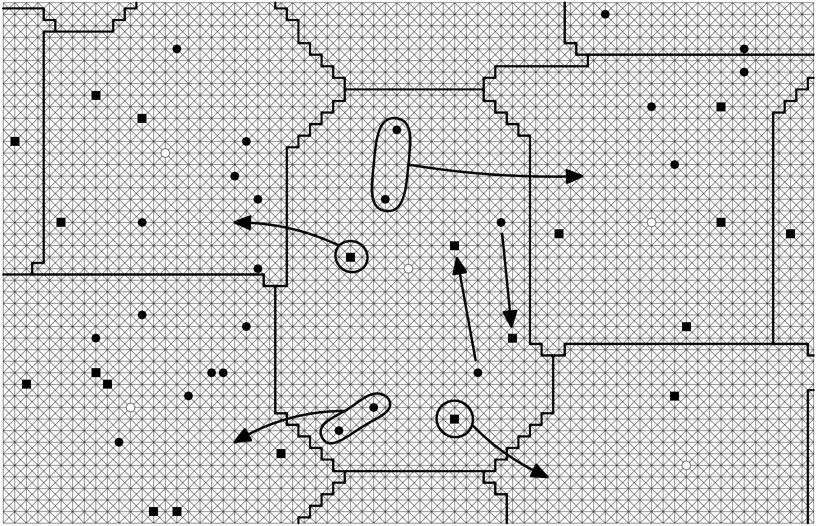
Bounded integer-valued AB-flow $\Rightarrow$ $A \sim B$



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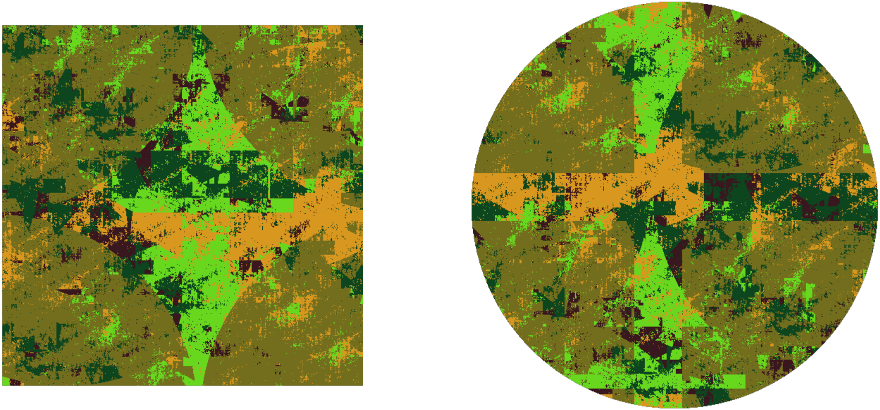
- Analogous Borel results for e.g. $SO(3) \curvearrowright \mathbb{S}^2$?

Discrete circle squaring (by András Máthé)



580 × 580 torus, 5 pieces, working modulo 1

Thank you!